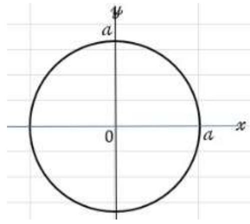
1

円　$x^2+y^2=a^2$　の全周



$$2x+2y\frac{dy}{dx}=0 \qquad \text{だから} \qquad \frac{dy}{dx}=-\frac{x}{y}$$

$$\mathcal{L}=4\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\int_0^a\sqrt{1+\frac{x^2}{a^2-x^2}}\,dx=4\int_0^a\sqrt{\frac{a^2}{a^2-x^2}}\,dx$$

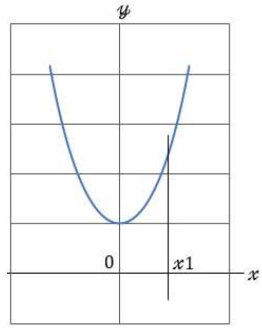
$$=4a\int_0^a\frac{1}{\sqrt{a^2-x^2}}\,dx=4a\left[\sin^{-1}\frac{x}{a}\right]_0^a=4a\frac{\pi}{2}=2a\pi$$

2

懸垂戦　$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$　が　y軸を通る点から、

y軸の右の方ににある

曲線上の点（$x1$，$y12$）　までの弧の長さ



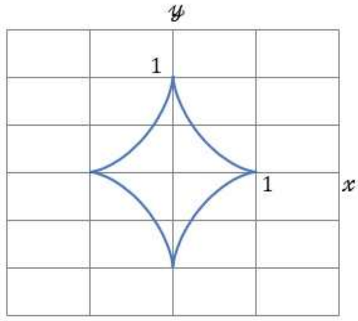
$$\frac{dy}{dx}=\frac{a}{2}\left(\frac{1}{a}e^{\frac{x}{a}}-\frac{1}{a}e^{-\frac{x}{a}-}\right)=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$$

$$\mathcal{L}=\int_0^{x_1}\sqrt{1+\frac{1}{4}\left(e^{\frac{2x}{a}}-2+e^{\frac{-2x}{a}}\right)}\,dx=\frac{1}{2}\int_0^{x_1}\sqrt{e^{\frac{2x}{a}}+2+e^{\frac{-2x}{a}}}\,dx=\frac{1}{2}\int_0^{x_1}\sqrt{\left(e^{\frac{x}{a}}+e^{\frac{-x}{a}}\right)^2}\,dx$$

$$=\frac{1}{2}\left[ae^{\frac{x}{a}}-ae^{\frac{-x}{a}}\right]_0^{x_1}=\frac{a}{2}\left(e^{\frac{x_1}{2}}-e^{\frac{-x_1}{a}}\right)$$

3

アストロイド　$x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$　の全長



微分すると

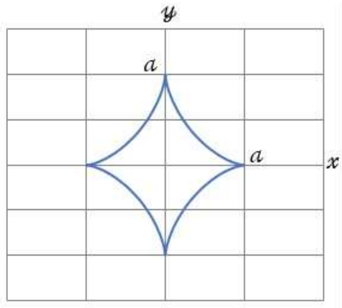
$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0 \qquad \text{だから} \qquad \frac{dy}{dx}=\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$\mathcal{L}=4\int_0^a\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\int_0^a\sqrt{1+\frac{a^{\frac{2}{3}}-x^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,dx=4\int_0^a\sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\,dx=4a^{\frac{1}{3}}\int_0^a x^{-\frac{1}{3}}\,dx$$

$$=4a^{\frac{1}{3}}\left[\frac{3}{2}x^{\frac{2}{3}}\right]_0^a=4a^{\frac{1}{3}}\cdot\frac{3}{2}a^{\frac{2}{3}}=6a$$

4

曲線(アストロイド) $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ の全長



微分すると

$$\left(\frac{1}{a}\right)^{\frac{2}{3}}\cdot\frac{2}{3}x^{-\frac{1}{3}}+\left(\frac{1}{b}\right)^{\frac{2}{3}}\cdot y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

となるから $\frac{dy}{dx}=-\left(\frac{b}{a}\right)^{\frac{2}{3}}\left(\frac{y}{x}\right)^{\frac{1}{3}}$

$$\mathcal{L}=4\int_0^a\sqrt{1+\left(\frac{b}{a}\right)^{\frac{4}{3}}\left(\frac{y}{x}\right)^{\frac{2}{3}}}dx=4\int_0^a\sqrt{1+\left(\frac{b}{a}\right)^2\frac{\left(\frac{y}{b}\right)^{\frac{2}{3}}}{\left(\frac{x}{a}\right)^{\frac{2}{3}}}}dx$$

$$=4\int_0^a\sqrt{\frac{a^2\left(\frac{x}{a}\right)^{\frac{2}{3}}+b^2\left\{1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right\}}{a^2\left(\frac{x}{a}\right)^{\frac{2}{3}}}}dx=4\int_0^a\sqrt{\frac{(a^2-b^2)\left(\frac{x}{a}\right)^{\frac{2}{3}}+b^2}{a^2\left(\frac{x}{a}\right)^{\frac{2}{3}}}}dx$$

$\left(\frac{x}{a}\right)^{\frac{2}{3}}=t$ とおくと $x=at^{\frac{3}{2}}$ $dx=\frac{3a}{2}\cdot t^{\frac{1}{2}}dt$

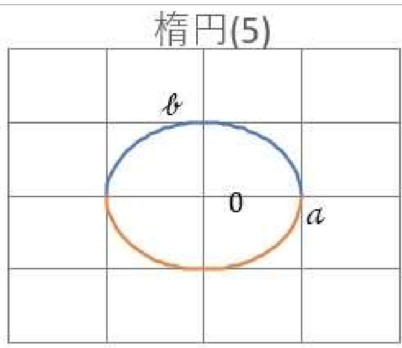
$$=4\int_0^1\sqrt{\frac{(a^2-b^2)t+b^2}{a^2t}}\cdot\frac{3}{2}at^{\frac{1}{2}}dt=6\int_0^1\sqrt{(a^2-b^2)t+b^2}\,dt$$

$$=6\left[\frac{2}{3}\{(a^2-b^2)t+b^2\}^{\frac{3}{2}}\cdot\frac{1}{a^2-b^2}\right]_0^1=\frac{4}{a^2-b^2}(a^3-b^3)$$

$$=\frac{4(a-b)(a^2+ab+b^2)}{(a+b)(a-b)}=\frac{4(a^2+ab+b^2)}{a+b}$$

5

楕円 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ の周の長さ



微分すると $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$

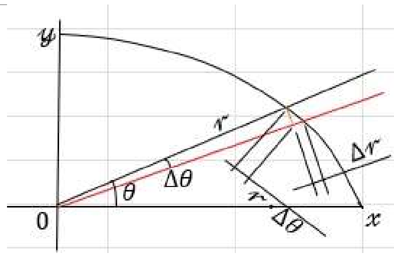
となるから $\frac{dy}{dx}=-\frac{b\left(\frac{x}{a}\right)}{a\left(\frac{y}{b}\right)}$

$$\mathcal{L}=4\int_0^a\sqrt{1+\frac{b^2}{a^2}\cdot\frac{\frac{x^2}{a^2}}{\frac{y^2}{b^2}}}dx=4\int_0^a\sqrt{1+\frac{b^2}{a^2}\cdot\frac{\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}=4\int_0^a\sqrt{\frac{a^2\left(1-\frac{x^2}{a^2}\right)+b^2\frac{x^2}{a^2}}{a^2\left(1-\frac{x^2}{a^2}\right)}}$$

この定積分は楕円積分というものであって、初等関数では積分不可である

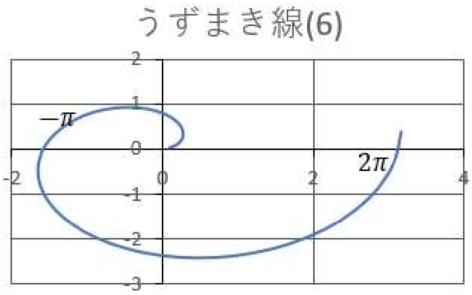
6	参考公式

$$\int_0^{\pi}\sqrt{(r\cdot\Delta\theta)^2+(\Delta r)^2}=\int_0^{\pi}\sqrt{\left\{r^2+\left(\frac{\Delta r}{\Delta\theta}\right)^2\right\}^2(\Delta\theta)^2}$$



うずまき線　$r=a\theta$　$(a>0)$　の　$\theta=0$　から

$\theta=\psi$　までの曲線の長さ



$$\int_0^{\psi}\sqrt{(r\cdot\Delta\theta)^2+(\Delta r)^2}=\int_0^{\psi}\sqrt{\left\{r^2+\left(\frac{\Delta r}{\Delta\theta}\right)^2\right\}^2(\Delta\theta)^2}$$

求める全長は

$$\mathcal{L}=\int_0^{\psi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta=\int_0^{\psi}\sqrt{a^2\theta^2+a^2}d\theta=a\int_0^{\psi}\sqrt{\theta^2+1}d\theta$$

$$=\frac{a}{2}\left[\theta\sqrt{\theta^2+1}+\log\left(\theta+\sqrt{\theta^2+1}\right)\right]_0^{\psi}=\frac{a}{2}\left(\psi\sqrt{\psi^2+1}+\log\left(\psi+\sqrt{\psi^2+1}\right)\right)$$