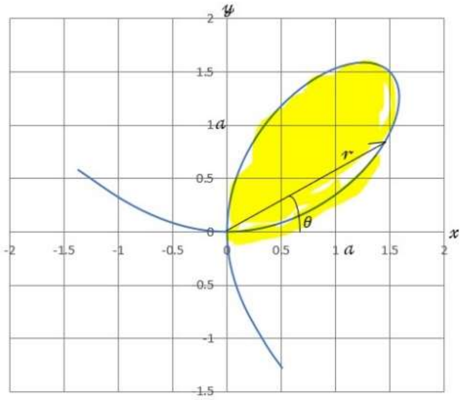
7

正葉曲線　$x^3 + y^3 - 3axy = 0$　の自閉線内の面積




$x = r\cos\theta$　　$y = r\sin\theta$　だから

$r^3\cos^3\theta + r^3\sin^3\theta = 3ar^2\cos\theta\sin\theta$

$$r = \frac{3a\cos\theta\sin\theta}{\cos^2\theta + \sin^2\theta}$$

したがって求める面積は

$$S = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{3a\cos\theta\sin\theta}{\cos^3\vartheta + \sin^3\theta}\right)^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{3a\tan\theta\sec\theta}{1+\tan^3\theta}\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{(3a)^2\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}d\theta$$

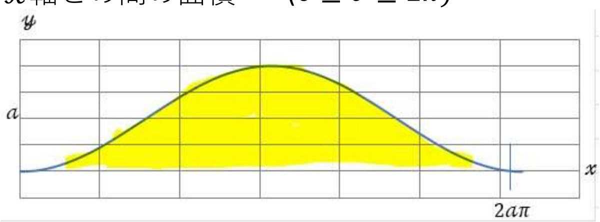
$\tan^3\theta = \tau$　とおくと　$3\tan^2\theta\sec^2\theta = dt$

$$= \frac{1}{2}\int_0^{\infty}\frac{(3a)^2\tan^2\theta\sec^2\theta}{(1+t)^2}\cdot\frac{dt}{3\tan^2\theta\sec^2\theta} = \frac{1}{2}\int_0^{\infty}\frac{3a^2}{(1+t)^2}dt = \frac{3}{2}a^2\left[-\frac{1}{1+t}\right]_0^{\infty}$$

$$= \frac{3}{2}a^2(0+1) = \frac{3}{2}a^2$$

8

曲線　$x = a\theta$　、$y = a(1-\cos\theta)$　と　$x$軸との間の面積　　$(0 \leqq \theta \leqq 2\pi)$




$y = a(1-\cos\theta) = 0$　とすると

$\theta = 0$　$2\pi$

$x = 0$　$2a\pi$

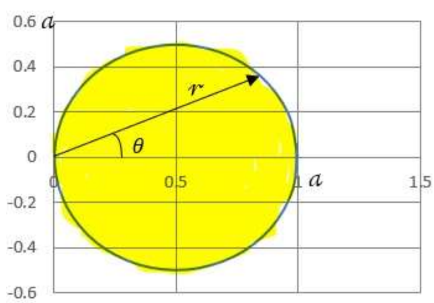
したがって求める面積は $dx = ad\theta$　であるので

$$S = \int_0^{2a\pi} y\,dx = \int_0^{2\pi} a(1-\cos\theta)a\,d\theta = a^2\int_0^{2\pi}(1-\cos\theta)d\theta$$

$$= a^2[\theta - \sin\theta]_0^{2\pi} = 2a^2\pi$$

9

円　$r = a\cos\theta$　の面積




$$S = 2\int_0^{\frac{\pi}{2}}\frac{1}{2}r^2 d\theta = \int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta$$

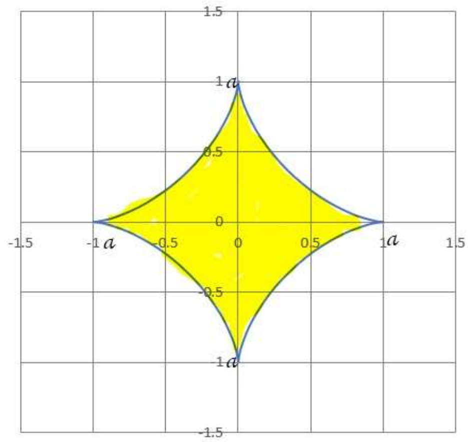
$$= a^2\int_0^{\frac{\pi}{2}}\frac{1}{2}(\cos 2\theta + 1)d\theta$$

$$= a^2\left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{a^2}{4}\pi$$

10 アストロイド　$x = a\cos^3\theta$　、$y = a\sin^3\theta$　のつつむ面積

$y=0$　のとき　$\theta=0$　だから　$x=a$

したがって求める面積は




$$S = 4\int_0^a y\,dx = 4\int_{\frac{\pi}{2}}^0 y\frac{dx}{d\theta}d\theta$$

$$= 4\int_{\frac{\pi}{2}}^0 a\sin^3\theta * 3a\cos^2\theta * (-\sin\theta)d\theta$$

$$= 12a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^2\theta d\theta$$

※（公式26、27，参照）

公式26　$$\mathcal{L}_{m,n} = \frac{\sin^{m+1}x\cdot\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\mathcal{K}_{m,n-2}$$

公式27　$$\mathcal{L}_{m,n} = -\frac{\sin^{m-1}x\cdot\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\mathcal{K}_{m-2,n}$$

$$\int \sin^4\theta\cos^2\theta d\theta = -\frac{\sin^3\theta\cos^3\theta}{6} + \frac{3}{6}\int \sin^2\theta\cos^2\theta d\theta$$

$$= -\frac{1}{6}\sin^3\theta\cos^3\theta + \frac{1}{2}\left(-\frac{\sin\theta\cos^3\vartheta}{4} + \frac{1}{4}\int \cos^2\theta d\vartheta\right)$$

$$= -\frac{1}{6}\sin^3\theta\cos^3\theta - \frac{1}{8}\sin\theta\cos^3\theta + \frac{1}{8}\int \frac{1}{2}(\cos 2\theta + 1)d\theta$$

したがって

$$※ = 12a^2\left[-\frac{1}{6}\sin^3\theta\cos^3\theta - \frac{1}{8}\sin\theta\cos^3\theta + \frac{1}{8}\left(\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right)\right]_0^{\frac{\pi}{2}}$$

$$= 12a^2\cdot\frac{1}{16}\cdot\frac{1}{2}\pi = \frac{3}{8}a^2\pi$$