APPENDIX A
EVALUATION OF $|v_i(r|\theta, l_i)|_3$

A. First case

The first case corresponds to the condition $C_1(\phi_i, l_i, r)$: $3\pi/2 < \phi_i < 2\pi, r < l_i|\sin\phi_i|$ (Fig. 2-(a)). In this case, the overlap occurs when $0 \le \theta \le \phi_i - \pi$. According to Fig. 2-(a), for $\phi_i - \pi < \theta < \pi$,

$$|v_i(r|\theta, l_i)|_2 = 0, \tag{20}$$

and, for $0 \le \theta \le \phi_i - \pi$,

$$|v_i(r|\theta, l_i)|_2 = \frac{r\sin\tau_i}{2}(r\cos\tau_i + \frac{r\sin\tau_i}{\tan(2\pi - \phi_i)}), \tag{21}$$

where $\tau_i \stackrel{\text{def}}{=} \phi_i - \theta - \pi$.

B. Second case

The second case corresponds to the condition $C_2(\phi_i, l_i, r)$: $\pi < \phi_i \le 3\pi/2, r < l_i$ (Fig. 2-(b)). According to Fig. 2-(b), for $\phi_i - \pi < \theta < \pi$,

$$|v_i(r|\theta, l_i)|_2 = 0, \tag{22}$$

and for $0 \le \theta \le \phi_i - \pi$,

$$|v_i(r|\theta, l_i)|_2 = \frac{r\sin\tau_i}{2}(r\cos\tau_i - \frac{r\sin\tau_i}{\tan(\phi_i - \pi)}). \tag{23}$$

Because $\tan(2\pi - \phi_i) = -\tan(\phi_i) = -\tan(\phi_i - \pi)$, the equations for the second case become identical to those for the first case. $C_{12}(\phi_i, l_i, r)$ is defined in Subsection IV-B.

C. Third case

The third case corresponds to the condition $C_3(\phi_i, l_i, r)$: $3\pi/2 < \phi_i < 2\pi, l_i|\sin\phi_i| \le r$ (Fig. 2-(c)). In this case, we need to consider three sub-cases. When two vertices on the diagonal of the parallelogram $\omega(r|\theta, l_i)$ are on the line on which the $(i+1)$-th line segment lies, $l_i\sin(2\pi - \phi_i) = r\sin(\phi_i - \pi - \theta)$. Let $\zeta_1(\phi_i, l_i, r), \zeta_2(\phi_i, l_i, r)$ be θ satisfying $l_i\sin(2\pi - \phi_i) = r\sin(\phi_i - \pi - \theta)$ (Fig. 2-(c')). Here, $\zeta_1(\phi_i, l_i, r)$, $\zeta_2(\phi_i, l_i, r)(\le \zeta_1(\phi_i, l_i, r))$, and $\gamma(\phi_i, l_i, r) > 0$ are defined in Subsection IV-B, where $\arcsin$ takes a value between $-\pi/2$ and $\pi/2$.

Dependent on whether θ is larger than $\zeta_1(\phi_i, l_i, r)$ ($\zeta_2(\phi_i, l_i, r)$), the shape of $v_i(r|\theta, l_i)$ is different: triangle or quadrangle. Therefore, we obtain the following. For $\phi_i - \pi < \theta < \pi$,

$$|v_i(r|\theta, l_i)|_2 = 0, \tag{24}$$

for $\zeta_1(\phi_i, l_i, r) \le \theta \le \phi_i - \pi$ or for $0 < \theta \le [\zeta_2(\phi_i, l_i, r)]^+$,

$$|v_i(r|\theta, l_i)|_2 = \frac{r\sin\tau_i}{2}(r\cos\tau_i + \frac{r\sin\tau_i}{\tan(2\pi - \phi_i)}), \tag{25}$$

and for $[\zeta_2(\phi_i, l_i, r)]^+ < \theta \le \zeta_1(\phi_i, l_i, r)$,

$$|v_i(r|\theta, l_i)|_2 = l_i r\sin\theta - \frac{l_i\sin(2\pi - \phi_i)}{2}(l_i\cos(2\pi - \phi_i) + \frac{l_i\sin(2\pi - \phi_i)}{\tan\tau_i}). \tag{26}$$

D. Fourth case

The fourth case corresponds to the condition $C_4(\phi_i, l_i, r)$: $\pi < \phi_i \le 3\pi/2, l_i \le r$ (Fig. 2-(d)). In this case, similarly to the third case, we need to consider three sub-cases. Because $\sin(2\pi - \phi_i) = \sin(\phi - \pi)$ and $l_i \le r$, $0 < \zeta_1(\phi_i, l_i, r) < \phi_i - \pi$. In addition, $\zeta_2(\phi_i, l_i, r) < 0$. Then, for $\phi_i - \pi < \theta < \pi$,

$$|v_i(r|\theta, l_i)|_2 = 0, \tag{27}$$

for $\zeta_1(\phi_i, l_i, r) \le \theta \le \phi_i - \pi$,

$$|v_i(r|\theta, l_i)|_2 = \frac{r\sin\tau_i}{2}(r\cos\tau_i + \frac{r\sin\tau_i}{\tan(2\pi - \phi_i)}), \tag{28}$$

and for $0 < \theta \le \zeta_1(\phi_i, l_i, r)$,

$$|v_i(r|\theta, l_i)|_2 = l_i r\sin\theta - \frac{l_i\sin(2\pi - \phi_i)}{2}(l_i\cos(2\pi - \phi_i) + \frac{l_i\sin(2\pi - \phi_i)}{\tan\tau_i}). \tag{29}$$

Let $C_{34}(\phi_i, l_i, r) \stackrel{\text{def}}{=} C_3(\phi_i, l_i, r) \cup C_4(\phi_i, l_i, r)$.

E. Calculation of $|v_i(r|\theta, l_i)|_3$

Note that $|v_i(r|\theta, l_i)|_2$ is independent of the rotation of the x-axis, we can calculate it by assuming that the x-axis is set along the i-th line segment. By using $|v_i(r|\theta, l_i)|_2$ under this assumption, we can calculate $|v_i(r|\theta, l_i)|_3 = \int |v_i(r|\theta, l_i)|_2 d\theta$.

For each concave ϕ_i, overlap $v_i(r|\theta, l_i)$ may occur between T and $\omega(r|\theta, l_i)$. In addition, the concave ϕ_i may cause the other overlap $v_i(r|\theta, l_{i+1})$ between T and $\omega(r|\theta, l_{i+1})$. Due to Eq. (7),

$$|T_s(r)|_3 = 2r|T|_1 - \sum_i \sum_{j=12,3,4} \sum_{l=l_i,l_{i+1}} \mathbf{1}(C_j(\phi_i, l, r)) \int_0^\pi |v_i(r|\theta, l)|_2 d\theta. \tag{30}$$

Here,

$$\begin{aligned} & \mathbf{1}(C_{12}(\phi_i, l, r)) \int_0^\pi |v_i(r|\theta, l)|_2 d\theta \\ = \; & \mathbf{1}(C_{12}(\phi_i, l, r))\{\int_0^{\phi_i - \pi} \frac{r \sin \tau_i}{2}(r \cos \tau_i - \frac{r \sin \tau_i}{\tan \phi_i}) d\theta\} \\ = \; & -\mathbf{1}(C_{12}(\phi_i, l, r)) r^2 g_1(\phi_i)/8 \\ = \; & \mathbf{1}(C_{12}(\phi_i, l, r)) g_{12}(\phi_i, r) \end{aligned} \tag{31}$$

where $g_1(\phi_i), g_{12}(\phi_i, r)$ are defined in Section IV-B. This is because

$$\begin{aligned} & \int_a^b \frac{r \sin \tau_i}{2}(r \cos \tau_i - \frac{r \sin \tau_i}{\tan \phi_i}) d\theta \\ = \; & [-\frac{r^2}{8}(-\cos(2\tau_i) + \frac{-2\tau_i + \sin(2\tau_i)}{\tan \phi_i})]_{\tau_i = \phi_i - a - \pi}^{\phi_i - b - \pi}. \end{aligned} \tag{32}$$

In addition,

$$\begin{aligned} & \mathbf{1}(C_3(\phi_i, l, r)) \int_0^\pi |v_i(r|\theta, l)|_2 d\theta \\ = \; & \mathbf{1}(C_3(\phi_i, l, r))\{\int_{\zeta_1(\phi_i, l, r)}^{\phi_i - \pi} \frac{r \sin \tau_i}{2}(r \cos \tau_i - \frac{r \sin \tau_i}{\tan \phi_i}) d\theta \\ & + \int_0^{[\zeta_2(\phi_i, l, r)]^+} \frac{r \sin \tau_i}{2}(r \cos \tau_i - \frac{r \sin \tau_i}{\tan \phi_i}) d\theta \\ & + \int_{[\zeta_2(\phi_i, l, r)]^+}^{\zeta_1(\phi_i, l, r)} lr \sin \theta + \frac{l \sin \phi_i}{2}(l \cos \phi_i - \frac{l \sin \phi_i}{\tan \tau_i}) d\theta\}, \\ = \; & \mathbf{1}(C_3(\phi_i, l, r)) g_3(\phi_i, l, r), \end{aligned} \tag{33}$$

where $g_{3,1}(\phi_i, l, r)$, $g_{3,2}(\phi_i, l, r)$, $g_{3,3}(\phi_i, l, r)$, $g_{3,4}(\phi_i, l, r)$ and $g_3(\phi_i, l, r)$ are defined in Secction IV-B. This is due to Eq. (32) and because, for $j = 1, 2$,

$$\cos(2(\phi_i - \pi - \zeta_j(\phi_i, l_i, r))) = 1 - 2\gamma(\phi_i, l_i, r)^2, \tag{34}$$

$$\begin{aligned} & \sin(2(\phi_i - \pi - \zeta_j(\phi_i, l_i, r))) \\ & = \begin{cases} 2\gamma(\phi_i, l_i, r)\sqrt{1 - \gamma(\phi_i, l_i, r)^2}, & \text{for } j = 1, \\ -2\gamma(\phi_i, l_i, r)\sqrt{1 - \gamma(\phi_i, l_i, r)^2}, & \text{for } j = 2. \end{cases} \end{aligned} \tag{35}$$

Furthermore,

$$\begin{aligned} & \mathbf{1}(C_4(\phi_i, l, r)) \int_0^\pi |v_i(r|\theta, l)|_2 d\theta \\ = \; & \mathbf{1}(C_4(\phi_i, l, r))\{\int_{\zeta_1(\phi_i, l, r)}^{\phi_i - \pi} \frac{r \sin \tau_i}{2}(r \cos \tau_i - \frac{r \sin \tau_i}{\tan \phi_i}) d\theta \\ & + \int_0^{\zeta_1(\phi_i, l, r)} lr \sin \theta + \frac{l \sin \phi_i}{2}(l \cos \phi_i - \frac{l \sin \phi_i}{\tan \tau_i}) d\theta\}, \\ = \; & \mathbf{1}(C_4(\phi_i, l, r)) g_4(\phi_i, l, r). \end{aligned} \tag{36}$$

where $g_4(\phi_i, l, r)$ is defined in Section IV-B.

APPENDIX B
PROOF OF THEOREM 6

Note that I_k has the following characteristic.

Lemma 1: Assume the sufficient condition in Theorem 6. When $s^\dagger \prec I_k \prec s' \prec I_{k+1}$, there is only a single pair $(\phi, l) \in \{(\phi_i, l_i), (\phi_i, l_{i+1})\}_i$ such that

$$\begin{array}{rcl} \mathbf{1}(C_{12}(\phi, l, s^\dagger)) & = & 1, \\ \mathbf{1}(C_{34}(\phi, l, s^\dagger)) & = & 0, \\ \mathbf{1}(C_{12}(\phi, l, s')) & = & 0, \\ \mathbf{1}(C_{34}(\phi, l, s')) & = & 1. \end{array}$$

This lemma is clear due to the definitions of C_{12}, C_{34}, and $\{I_k\}_k$.

Lemma 2: When $r = l + \epsilon$,

$$\sum_{k=3,4} \mathbf{1}(C_k(\phi, l, r)) g_k(\phi, l, r) \ = \mathbf{1}(C_{34}(\phi, l, r)) g_4(\phi, l, r) \tag{37}$$

where ϵ is a very small positive constant.

Proof: When $\mathbf{1}(C_3(\phi, l, l+\epsilon)) = 0$ and $\mathbf{1}(C_4(\phi, l, l+\epsilon)) = 1$, it is clear that Eq. (37) is valid.

Assume that $\mathbf{1}(C_3(\phi, l, l+\epsilon)) = 1$ and $\mathbf{1}(C_4(\phi, l, l+\epsilon)) = 0$. This means $3\pi/2 < \phi < 2\pi$. Then, $\arcsin \gamma(\phi, l, l+\epsilon) = 2\pi - \phi - \epsilon'$ where ϵ' is a very small positive constant. Thus, $\zeta_2(\phi, l, l+\epsilon) = -\epsilon' < 0$. When $\zeta_2(\phi, l, l+\epsilon) < 0$, $g_3(\phi, l, l+\epsilon) = g_4(\phi, l, l+\epsilon)$. Therefore, Eq. (37) is valid. □

A. Main part of proof of Theorem 6

When $\phi_i, l_i \in \Xi_k$ and $l_{i+1} \notin \Xi_k$ ($\phi_i, l_{i+1} \in \Xi_k$ and $l_i \notin \Xi_k$), we call l_{i+1} (l_i) the undetermined line segment of (ϕ_i, Ξ_k) and use the notation $l(\phi_i, \Xi_k)$ to express this undetermined line segment. When $I_k = [l_i \langle \sin\phi_i \rangle, l_i]$, define $l(I_k) = l_i$. Set $\Xi_1 = \emptyset$.

For $s' \prec I_1$, $\sum_{l=l_j, l_{j+1}} \mathbf{1}(C_{34}(\phi_j, l, s')) = 0$ for any j. Therefore, due to Lemma 1, for $I_1 \prec s_3 < s_4 \prec I_2$, there exists a single pair (ϕ_i, l) that satisfies the following conditions with $k = 3, 4$ where $l = l_i$ or l_{i+1}:

$$\begin{cases} \mathbf{1}(C_{34}(\phi_i, l, s_k)) > 0, \\ \mathbf{1}(C_{34}(\phi_j, l, s_k)) = 0, \quad \text{for any } j \neq i. \end{cases}$$

We can assume $\mathbf{1}(C_{34}(\phi_i, l_i, s_k)) = 1, \mathbf{1}(C_{34}(\phi_i, l_{i+1}, s_k)) = 0$ without loss of generality. Set $s_k - l(I_1) = \epsilon_k$, a very small positive constant, for $k = 3, 4$, and apply Eq. (37) with $l = l(I_1), r = l(I_1) + \epsilon_k$. Then, due to Eq. (12), we can obtain

$$q(s_k) = -g_4(\phi_i, l_i, s_k) + g_{12}(\phi_i, s_k)$$

($k = 3, 4$). By using a given $q(s_3)$ and $q(s_4)$, unknown parameters ϕ_i and l_i are uniquely determined (see Appendix C-A). Update Ξ_0 by adding the parameters determined here. That is, set $\Xi_1 = \Xi_0 \cup \{\phi_i, l_i\} = \{\phi_i, l_i\}$.

Similarly, for $I_2 \prec r \prec I_3$, we can determine a pair $(\phi, l) \notin \Xi_1$ or a undetermined line segment $l(\phi, \Xi_1)$. Note, for $I_2 \prec r \prec I_3$, (1) or (2) defined below occurs:

(1) there exists a single pair $(\phi, l) \notin \Xi_1$ that

$$\begin{cases} \sum_l \mathbf{1}(C_{34}(\phi, l, r)) > 0, \\ \sum_l \mathbf{1}(C_{34}(\phi', l, r)) = 0, \quad \text{for any } \phi' \notin \Xi_1 \cup \{\phi\}, \end{cases}$$

(2) there exists $(\phi, l) \in \Xi_1$ that

$$\begin{cases} \mathbf{1}(C_{34}(\phi, l, r)) = 1, \\ \mathbf{1}(C_{34}(\phi, l(\phi, \Xi_1), r)) = 1, \\ \sum_{l=l_j, l_{j+1}} \mathbf{1}(C_{34}(\phi_j, l, r)) = 0, \quad \text{for any } \phi_j \notin \Xi_1. \end{cases}$$

For (1), we can assume $\mathbf{1}(C_{34}(\phi_k, l_k, r)) = 1$, $\mathbf{1}(C_{34}(\phi_k, l_{k+1}, r)) = 0$ without loss of generality. Then, for $I_2 \prec s_5 < s_6 \prec I_3$, set $s_j - l(I_2)$ to be a very small positive constant ϵ_j for $j = 5, 6$, and apply Eq. (37) with $l = l(I_2), r = l(I_2) + \epsilon_j$. Then, due to Eq. (12),

$$q(s_j) + \sum_{m=3}^{4} \sum_{\phi, l \in \Xi_1} \mathbf{1}(C_m(\phi, l, s_j)) g_m(\phi, l, s_j) - g_{12}(\phi, s_j) = -g_4(\phi_k, l_k, s_j) + g_{12}(\phi_k, s_j),$$

where $j = 5, 6$. Because we have already uniquely determined ϕ_i, l_i, the left-hand side of the above equation is given. Thus, ϕ_k, l_k are uniquely determined (see Appendix C-A). Update Ξ_1 by adding the parameters determined here. That is, set $\Xi_2 = \Xi_1 \cup \{\phi_k, l_k\}$.

For (2), for $I_2 \prec s_5 \prec I_3$, set $s_5 - l(I_2)$ to be a very small positive constant ϵ_5 and apply Eq. (37) with $l = l(I_2), r = l(I_2) + \epsilon_5$. Then, due to Eq. (12),

$$q(s_5) + \sum_{m=3}^{4} \sum_{\phi, l \in \Xi_1} \mathbf{1}(C_m(\phi, l, s_5)) g_m(\phi, l, s_5) - 2g_{12}(\phi, s_5) = -g_4(\phi_i, l(\phi_i, \Xi_1), s_5).$$

By using the given left-hand side of the above equation, $l(\phi_i, \Xi_1)$ is uniquely determined (see Appendix C-B). Update Ξ_1 by adding the parameters determined here. That is, set $\Xi_2 = \Xi_1 \cup \{l(\phi_i, \Xi_1)\}$.

By repeating this procedure, we can uniquely determine $|T|_1$, $|T|_2$, and $\{l_i, l_{i+1}, \phi_i > \pi\}_i$.

APPENDIX C
PROOF OF UNIQUENESS

A. *Proof of uniqueness regarding ϕ and l*

In this section, we use the notations

$$\mathcal{G}_1 \equiv \Big\{(\phi, l, r) | \pi < \phi \le 3\pi/2,\ l \le r\Big\},$$
$$\mathcal{G}_2 \equiv \Big\{(\phi, l, r) | 3\pi/2 < \phi < 2\pi,\ l|\sin\phi| \le r\Big\}.$$

Below, we define $\arcsin(\cdot)$ with its value on the interval $(-\pi/2, \pi/2)$ and $\arccos(\cdot)$ on $(0, \pi)$. We also simplify the representations of some of the functions below. They were obtained with the aid of elementary calculations, and we omit the proof:

$$\begin{cases} \cos(\zeta_1(l,\phi,r)) = -\sqrt{1-\big(\gamma(\phi,l,r)\big)^2}\cos\phi \\ \qquad\qquad +lr^{-1}\sin^2\phi, \\ g_{3,1}(l,\phi,r) = -2\Big(\dfrac{l\sin\phi}{r}\Big)^2 + \dfrac{2\arcsin\gamma(l,\phi,r)}{\tan\phi} \\ \qquad\qquad +2lr^{-1}\sqrt{1-\big(\gamma(\phi,l;r)\big)^2}\cos\phi, \\ g_1(\phi) = -2\Big\{1 - \dfrac{(\phi-\pi)}{\tan\phi}\Big\}. \end{cases} \tag{38}$$

We prove the uniqueness of a solution when the estimability condition holds.

Using Eq. (38), we obtain the following equality.

$$\begin{aligned} \Phi(\phi,l,r) &\equiv -g_4(\phi,l,r) - \frac{r^2}{8}g_1(\phi) \\ &= -lr + \frac{r^2}{4} - \frac{3l}{4}\sqrt{r^2-(l\sin\phi)^2}\cos\phi \\ &\quad +\frac{3}{4}\big(l\sin\phi\big)^2 - \frac{r^2\zeta_1(\phi,l,r)}{4\tan\phi} \\ &\quad -\frac{l^2\zeta_1(\phi,l,r)}{2}\sin\phi\cos\phi \\ &\quad +\Big(\log(r/l)\Big)\frac{(l\sin\phi)^2}{2} \end{aligned} \tag{39}$$

Now, we introduce the notation $\boldsymbol{x} = (\phi, l)$ and regard r as a parameter for the meanwhile. We also use the notation $\Phi(\boldsymbol{x}; r) \equiv \Phi(l, \phi, r)$ hereafter. Then, we prove uniqueness in accordance with the following argument.

Let us suppose that two points $\{\boldsymbol{x}_j\}_{j=1,2} = \{(\phi_j, l_j)\}_{j=1}^2$ satisfy

$$\Phi(\boldsymbol{x}_1; r_i) = \Phi(\boldsymbol{x}_2; r_i) = d_i\ (i = 1, 2).$$

Since this means

$$\Phi(\boldsymbol{x}_1; r_i) - \Phi(\boldsymbol{x}_2; r_i) = 0\ (i = 1, 2),$$

by virtue of the mean value theorem, we have

$$(\boldsymbol{x}_1-\boldsymbol{x}_2)\cdot\int_0^1 \nabla\Phi(\eta\boldsymbol{x}_1+(1-\eta)\boldsymbol{x}_2;r_i)\,\mathrm{d}\eta=0 \qquad (i=1,2), \tag{40}$$

where $\nabla=(\partial/\partial\phi,\partial/\partial l)^{\mathrm{T}}$. Let us introduce

$$\begin{aligned}\tilde{\boldsymbol{x}}&=\boldsymbol{x}_1-\boldsymbol{x}_2,\\ f_1(r;\boldsymbol{x}_1,\boldsymbol{x}_2)&\equiv\int_0^1\frac{\partial\Phi}{\partial\phi}(\eta\boldsymbol{x}_1+(1-\eta)\boldsymbol{x}_2;r)\,\mathrm{d}\eta,\\ f_2(r;\boldsymbol{x}_1,\boldsymbol{x}_2)&\equiv\int_0^1\frac{\partial\Phi}{\partial l}(\eta\boldsymbol{x}_1+(1-\eta)\boldsymbol{x}_2;r)\,\mathrm{d}\eta.\end{aligned}$$

Then, Eq. (40) equals

$$\boldsymbol{M}(\boldsymbol{x}_1,\boldsymbol{x}_2)\tilde{\boldsymbol{x}}=\boldsymbol{0},$$

where

$$\boldsymbol{M}(\boldsymbol{x}_1,\boldsymbol{x}_2)=[m_{ij}]_{i,j=1,2}$$

with $m_{i,j}=f_j(r_i;\boldsymbol{x}_1,\boldsymbol{x}_2)$. Therefore, if

$$\det\boldsymbol{M}(\boldsymbol{x}_1,\boldsymbol{x}_2)\neq 0 \tag{41}$$

holds for every $(\boldsymbol{x}_1,\boldsymbol{x}_2)$, it means $\tilde{\boldsymbol{x}}=\boldsymbol{0}$, which directly leads to the desired statement.

Now, assume $r_1<r_2$ without loss of generality; then, Eq. (41) amounts to

$$f_1(r_1;\boldsymbol{x}_1,\boldsymbol{x}_2)f_2(r_2;\boldsymbol{x}_1,\boldsymbol{x}_2) -f_1(r_2;\boldsymbol{x}_1,\boldsymbol{x}_2)f_2(r_1;\boldsymbol{x}_1,\boldsymbol{x}_2)\neq 0. \tag{42}$$

Thereby, it is sufficient to show, for instance,

$$\begin{aligned}f_1(r_1;\boldsymbol{x}_1,\boldsymbol{x}_2)&\geq f_1(r_2;\boldsymbol{x}_1,\boldsymbol{x}_2),\\ f_2(r_1;\boldsymbol{x}_1,\boldsymbol{x}_2)&\leq f_1(r_2;\boldsymbol{x}_1,\boldsymbol{x}_2)\quad\forall\boldsymbol{x}_1,\boldsymbol{x}_2\end{aligned}$$

for Eq. (42). In particular, it suffices to show

$$\frac{\partial\Phi}{\partial l}(\phi,l,r_1)\geq\frac{\partial\Phi}{\partial l}(\phi,l,r_2), \tag{43}$$

$$\frac{\partial\Phi}{\partial\phi}(\phi,l,r_1)\leq\frac{\partial\Phi}{\partial\phi}(\phi,l,r_2) \tag{44}$$

$$\forall(\phi,l,r_i)\in\bigcup_{i=1}^{2}\mathcal{G}_i,\ r_1<r_2,$$

and we shall prove Eqs. (43)–(44) below.

It is obvious that $\Phi(\phi,l,r)$ is smooth enough with respect to (ϕ,l,r), and we first derive $\partial\Phi/\partial r$. By virtue of Eq. (39), after some calculations, we have

$$\frac{\partial\Phi}{\partial r}=-l+\frac{r}{2}-\frac{l\cos\phi}{2r}\sqrt{r^2-(l\sin\phi)^2}+\frac{(l\sin\phi)^2}{2r} -\frac{r}{2\tan\phi}\Big\{\phi-\pi+\arcsin(l\sin\phi/r)\Big\}. \tag{45}$$

From this, we deduce

$$\frac{\partial^2\Phi}{\partial l\partial r}=\frac{1}{r}\Big(-r+l\sin^2\phi-\sqrt{r^2-(l\sin\phi)^2}\cos\phi\Big). \tag{46}$$

Since $-r+l\sin^2\phi\leq 0$, it is sufficient to consider the case $\cos\phi\leq 0$ to show the non-positiveness of the right-hand side. It is easily seen that $|-r+l\sin^2\phi|\geq|\sqrt{r^2-(l\sin\phi)^2}\cos\phi|$ holds, since

$$\begin{aligned}|-r+l\sin^2\phi|^2 &- |\sqrt{r^2-(l\sin\phi)^2}\cos\phi|^2\\ &= (r-l)^2\sin^2\phi\geq 0.\end{aligned}$$

Thus, due to Eq. (46), $\frac{\partial^2\Phi}{\partial l\partial r}\leq 0$ holds for $\phi\in(\pi,2\pi)$, which corresponds to Eq. (43).

Next, using Eq. (45) again, some lengthy calculations yield

$$\frac{\partial^2\Phi}{\partial\phi\partial r} = \frac{1}{2r\sin^2\phi}\Bigg[2l^2\sin^3\phi\cos\phi - r^2\sin\phi\cos\phi + \sqrt{r^2-(l\sin\phi)^2}(2\sin^2\phi-1)l\sin\phi + r^2\Big\{\phi-\pi+\arcsin(l\sin\phi/r)\Big\}\Bigg]. \tag{47}$$

This time, we discuss the positivity of the right-hand side of Eq. (47) for two cases with respect to the values of ϕ: $(\phi, l, r) \in \mathcal{G}_1$ or $\mathcal{G}_2$.

The first case is when $(\phi, l, r) \in \mathcal{G}_1$, where $\cos\phi = -\sqrt{1-\sin^2\phi}$ holds. By taking into account $\phi-\pi = \arcsin(\sin(\phi-\pi)) = \arcsin(-\sin\phi)$, denoting $y = \sin\phi$ leads to

$$\frac{\partial^2\Phi}{\partial\phi\partial r} = (r^2 y - 2l^2y^3)\sqrt{1-y^2} + ly(2y^2-1)\sqrt{r^2-(ly)^2} + r^2\Big\{\arcsin(ly/r) - \arcsin(y)\Big\}, \tag{48}$$

where $y \in (-1, 0)$. To show the positivity of the right-hand side of Eq. (48), we regard it as a function of r by fixing the variable y, which is denoted by $f(r)$. Then,

$$g(r) \equiv \frac{f'(r)}{2r} = y\sqrt{1-y^2}\Big[1 - \frac{l\sqrt{1-y^2}}{\sqrt{r^2-(ly)^2}}\Big] + \Big\{\arcsin(ly/r) - \arcsin(y)\Big\}.$$

Obviously, $g(l) = 0$ holds, and in addition,

$$g'(r) = \frac{ly^3(l^2-r^2)}{r^2}\Big\{r^2-(ly)^2\Big\}^{-3/2} \geq 0 \quad \forall r \geq 0,\ y \in (-1, 0).$$

This implies $g(r) \geq 0\ \forall r \geq l$ for each $y \in (-1, 0)$, and therefore, $f'(r) \geq 0$ on the same interval. By noting that $f(l) = 0$ holds due to Eq. (48), we arrive at

$$\frac{\partial^2\Phi}{\partial\phi\partial r} \geq 0 \quad \forall r \geq 0,\ y \in (-1, 0).$$

In the second case, when $(\phi, l, r) \in \mathcal{G}_2$, we first note that

$$\begin{aligned} \phi - \pi \quad + & \quad \arcsin(l\sin\phi/r) \\ = & \quad \big(\phi - \frac{3\pi}{2}\big) + \frac{\pi}{2} + \arcsin(l\sin\phi/r) \\ = & \quad \arccos(-\sin\phi) + \arccos(-l\sin\phi/r). \end{aligned}$$

By applying this to Eq. (47) and introducing $z = -\sin\phi \in (0, 1)$, we obtain

$$\frac{\partial^2\Phi}{\partial\phi\partial r} = \frac{1}{2rz^2}\Bigg[(r^2z - 2l^2z^3)\sqrt{1-z^2} + rlz(1-2z^2)\sqrt{1-(lz/r)^2} + r^2\Big\{\arccos(z) + \arccos(lz/r)\Big\}\Bigg] \tag{49}$$

In accordance with the definition of $\arccos(z)$, we have

$$\begin{aligned}
&r^2 \arccos(z) + (r^2 z - 2l^2 z^3)\sqrt{1-z^2} \\
&\quad = 2z(r^2 - (lz)^2)\sqrt{1-z^2} + 2z^2 \int_z^1 \sqrt{1-t^2}\,\mathrm{d}t \geq 0, \\
&r^2 \arccos(lz/r) + (rlz - 2rlz^3)\sqrt{1-(lz/r)^2} \\
&\quad = 2rlz(1-z^2)\sqrt{1-(lz/r)^2} + 2r^2 \int_{lz/r}^1 \sqrt{1-t^2}\,\mathrm{d}t \\
&\quad \geq 0.
\end{aligned}$$

Thus, we arrive at

$$\frac{\partial^2 \Phi}{\partial\phi\partial r} \geq 0 \quad \forall r \geq 0$$

in this case also. This corresponds to Eq. (44), and now, we have completed the proof of uniqueness by virtue of the preceding arguments.

B. *Proof of uniqueness regarding l*

With (ϕ, r) provided, we solve the problem

$$g_4(l,\phi,r) = q \tag{50}$$

with respect to l. By making use of Eq. (38) in the preceding subsection, we write

$$\begin{aligned}
g_4(l,\phi,r) = lr&\left[1 + \frac{3}{4}\sqrt{1-\big(\gamma(\phi,l,r)\big)^2}\cos\phi\right] \\
&-\frac{3}{4}\big(l\sin\phi\big)^2 - \frac{r^2 \arcsin\gamma(\phi,l,r)}{4\tan\phi} \\
&+\frac{l^2\zeta_1(\phi)}{2}\sin\phi\cos\phi - \frac{(l\sin\phi)^2}{2}\log\Big(\frac{r}{l}\Big).
\end{aligned} \tag{51}$$

If we regard the right-hand side of Eq. (51) as a function of l and denote it by $\Psi(l)$, then we have

$$\begin{aligned}
\Psi'(l) = r &+ \sqrt{r^2 - \big(l\sin\phi\big)^2}\cos\phi \\
&-l\Big(1 + \log\Big(\frac{r}{l}\Big)\Big)\sin^2\phi \\
&+l\Big(\phi - \pi\arcsin\gamma(\phi,l,r)\Big)\sin\phi\cos\phi.
\end{aligned} \tag{52}$$

In the following, we show the non-negativity of $\Psi'(l)$ for any $(\phi,l,r) \in \mathcal{G}_i$ $(i=1,2)$ separately.

First, we consider the case $(\phi,l,r) \in \mathcal{G}_1$. In this case, we fix l and ϕ, and regard the right-hand of Eq. (52) as a function of r, denoted by $F(r)$. Then, we show $F(r) \geq 0\ \forall r \geq l$ with $(\phi,l,r) \in \mathcal{G}_1$. By virtue of elementary calculations, it is easily seen that

$$\begin{aligned}
F|_{r=l} &= J_1 + J_2, \\
J_1 &\equiv 2l\sin^2\phi, \\
J_2 &\equiv l\Big(\phi - \pi + \arcsin\Big(\frac{l\sin\phi}{r}\Big)\Big)\sin\phi\cos\phi.
\end{aligned}$$

It is obvious that $J_1 \geq 0$ holds. Then, introducing $y = \sin\phi$, we have $\phi - \pi = \arcsin(y)$ as we have seen in the previous subsection. This leads to

$$\phi - \pi + \arcsin\Big(\frac{l\sin\phi}{r}\Big) = \arcsin(ly/r) - \arcsin(y) \geq 0,$$

and we have $J_2 \geq 0$ on $\mathcal{G}_1$. These indicate

$$F|_{r=l} \geq 0. \tag{53}$$

In addition, it is easily seen that

$$F'(r) = 1 + \frac{\sqrt{r^2 - (l\sin\phi)^2}\cos\phi}{r} - \frac{l\sin^2\phi}{r} \geq 0.$$

This, together with Eq. (53), implies $\Psi'(l) \geq 0$ on $\mathcal{G}_1$.

Next, we consider the case $(\phi, l, r) \in \mathcal{G}_2$. Since $\phi \in (3\pi/2, 2\pi)$ in this case, we have

$$\begin{aligned}\phi - \pi + \arcsin\Big(\frac{l\sin\phi}{r}\Big) \\ = \arccos(-\sin\phi) + \arccos(-l\sin\phi/r).\end{aligned}$$

Therefore, by taking $z = -\sin\phi \in (0,1)$ and making use of $1 + \log x \leq x \ \forall x \geq 0$, we have

$$\begin{aligned}\Psi'(l) &= r + \sqrt{r^2 - (lz)^2}\sqrt{1-z^2} - lz^2\Big(1 + \log\Big(\frac{r}{l}\Big)\Big) \\ &\quad + lz\sqrt{1-z^2}\Big\{\arccos(z) + \arccos(lz/r)\Big\} \\ &\geq (r + lz^2)(1 - z^2) \\ &\quad + \Big(1 + \frac{(lz)^2}{r^2}\Big)\sqrt{r^2 - (lz)^2}\sqrt{1-z^2} \geq 0.\end{aligned}$$

This completes the proof.

Appendix D
Estimation method for $\{l_i, l_{i+1}, \phi_i\}$

Under limited cases such as a small number of concave vertices and large number of sensing results available, the following procedure theoretically can estimate $\{l_i, l_{i+1}, \phi_i\}$ when the estimability condition is satisfied. However, the estimated results are often unstable.

Let $\widehat{q(r)}$ be an estimator of $q(r)$. Then, it is given as follows.

$$\widehat{q(r)} = 2\pi|\Omega|_2 N(r)/n_s - 2\widehat{|T|_1} r - 2\pi\widehat{|T|_2} - 2r^2 \sum_i \widehat{g_1(\phi_i)}/8 \tag{54}$$

where $\sum_i \widehat{g_1(\phi_i)} = 4\sum_{j=1,2}\{\pi|\Omega|_2 N(s_j)/n_s - \widehat{|T|_1} s_j - \pi\widehat{|T|_2}\}/s_j^2$.

We expect that $q_1(r) \overset{\text{def}}{=} \widehat{q(r)} \approx 0$ for $r < I_1$. Therefore, by plotting $\widehat{q(r)}$, we can estimate where I_1 is and choose $s_3, s_4 > I_1$ (Fig. 10). s_3 and s_4 should not be too large, because they should satisfy $s_3, s_4 < I_2$. Because $q(s_k) = -g_4(\phi_i, l_i, s_k) - s_k^2 g_1(\phi_i)/8$ $(k = 3,4)$, we can estimate ϕ_i, l_i by using the calculated $q_1(s_3), q_1(s_4)$ and solving the following equations for $k = 3, 4$.

$$q_1(s_k) = -g_4(\phi_i, l_i, s_k) - s_k^2 g_1(\phi_i)/8 \tag{55}$$

By using the estimated ϕ_i, l_i, plot $q_{m+1}(r) \overset{\text{def}}{=} q_m(r) + \sum_{j=3,4}(g_j(\widehat{\phi_i}, \widehat{l_i}, r) + r^2 g_1(\widehat{\phi_i})/8)\mathbf{1}(C_j(\widehat{\phi_i}, \widehat{l_i}, r))$ with $m = 1$. We expect that $q_2(r) \approx 0$ for $r < I_2$. Therefore, by plotting $q_2(r)$, we can estimate where I_2 is and choose $s_5, s_6 > I_2$ (Fig. 10). For $I_2 < s_5 < s_6 < I_3$,

$$q_2(s_k) = -g_4(\phi_j, l_j, s_k) - s_k^2 g_1(\phi_k)/8 \tag{56}$$

for $k = 5, 6$. Hence, we can estimate ϕ_j, l_j by using the calculated $q_2(s_5), q_2(s_6)$ and solving the equations similar to Eq. (55) for $k = 5, 6$. If the estimated ϕ_j is approximately equal to ϕ_i, judge that $j = i$ and adopt the estimated l_j as l_{i+1}.

Repeat these steps by using $q_m(r)$. We expect that we can estimate $\{l_i, l_{i+1}, \phi_i\}_i$.

A numerical example is provided here to illustrate the estimation method mentioned above. T is that used in Subsection E. T has one concave vertex with $\phi_1 = 4.89$ and $l_1 = 9.434, l_2 = 5.385$. We performed a simulation with $N_d = 100000$ and $|\Omega|_2 = 2500$. According to Eq. (54), $q_1(r) = \widehat{q(r)}$ was obtained as a function of r (Fig. 10). On the basis of this figure, we chose $s_3 = 6, s_4 = 8$. Then, we obtained two equations (Eq. (55)) with $s_3 = 6, s_4 = 8$. As a solution of these equations, we obtained an estimated $\phi_1 = 5.40$ and $l_1 = 2.69$. By using these estimated results, we plotted $q_2(r)$ and chose $s_5 = 13, s_6 = 15$ (Fig. 10). We solved two equations (Eq. (56)) with $s_5 = 13, s_6 = 15$ and obtained the estimated $\phi_2 = 5.293$ and $l_2 = 10.79$. On the basis of the plotted $q_3(r)$ (Fig. 10), we judged that all $\{l_i, l_{i+1}, \phi_i\}_i$ were estimated because $q_3(r) \approx 0$ for $0 < \forall r < r_{max}$. Because $\widehat{\phi_1} \approx \widehat{\phi_2}$, we judge that they are identical.

Appendix E
Parameter design

With the proposed estimation method, we use parameters r_{max} and s_1, s_2. To obtain a good estimate, we should appropriately design the values of these parameters. We use a simple quadrangle as T and try various values of parameters in the estimation method to determine them. The quadrangle has four edges with $l_1 = 9.434, l_2 = 5.385, l_3 = 10$, and $l_4 = 10\sqrt{2}$ and one concave vertex with a radian of $\phi_1 = 4.89$.

We determine r_{max} by using $e(r)/\sigma$ for this quadrangle. We conducted a simulation, and the results are shown in Fig. 11 with $N_d \overset{\text{def}}{=} N_0 + N_+ = 10000$ for each simulation run. In this figure, the average of $e(r)/\sigma$ of 20 runs is plotted against

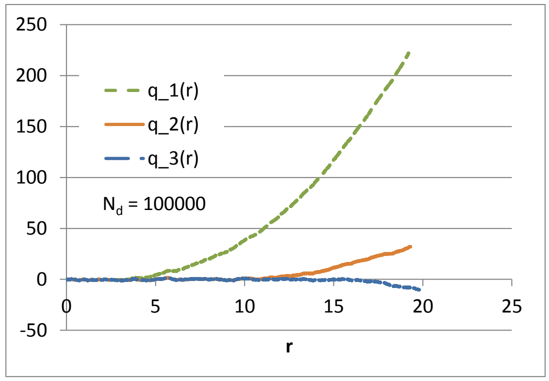

Fig. 10. Example of $q_1(r), q_2(r)$

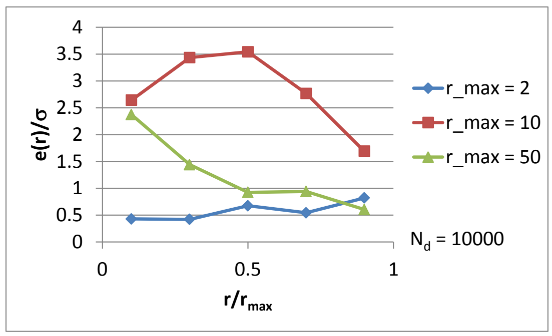

Fig. 11. $e(r)/\sigma$ vs. r_{max}

r/r_{max} for each r_{max}. According to this figure, $r_{max} \approx l_i$ or $r \approx l_i$ can detect non-convexity well if we judge it by evaluating that $e(r)/\sigma$ is larger than a threshold. Specifically, $r \approx \min(l_i, l_{i+1})$ is suitable to detect non-convexity.

However, we do not know the exact l_i. To be applicable to various T, it is better to use various r to cover various l_i. In the following, we use $r_{max} = 20$ and $r/r_{max} = 0.3, 0.5, 0.7$. We can then obtain three $e(r)/\sigma$ corresponding to each value of r/r_{max}. By using these three $e(r)/\sigma$, we choose the largest and judge the non-convexity of T if it is larger than a threshold.

Next, we determined s_1, s_2 by using the simulation results for this quadrangle with $N_d = 10000$. The results are shown in Fig. 12. For various s_1 and s_2, $|T|_1$ is estimated with the proposed method using Eqs. (14) and (16), and its relative error is evaluated. The last one is the result assuming the convexity and calculated using Eq. (14). As shown in this figure, $\widehat{|T|_1}$ assuming the convexity has a large negative bias, that is, underestimates $|T|_1$ for a non-convex T. However, the relative error has a very small variance. In comparison, $|T|_1$ estimated with the proposed method has a very small bias and a large variance. Therefore, the estimation of $|T|_1$ for a non-convex T requires a large number of samples. In general, (i) a too small or too large s_1 may cause a bias, and (ii) a large $s_2 - s_1$ has a small variance of the relative error. As a result, $s_1 = 1$ and $s_2 \approx l_i, l_{i+1}$ are appropriate combinations. Because we do not know the exact l_i, we calculate $\widehat{|T|_1}$ using Eq. (14) with a small number of samples and use, for example, $s_2 = \widehat{|T|_1}/10$.

Appendix F
T used in Subsection VII-A

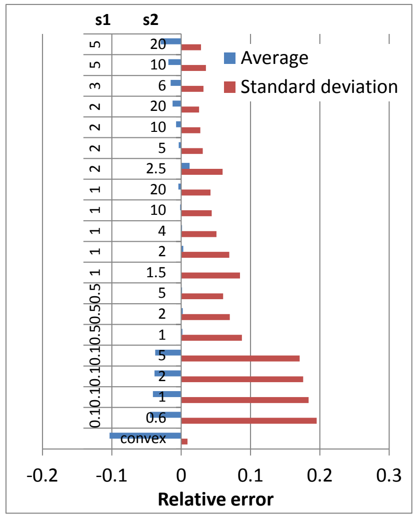

Fig. 12. Perimeter estimation errors with various s_1, s_2